

Associahedra, cellular W -construction and products of A_∞ -algebras

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Abstract. The aim of this paper is to construct a functorial tensor product of A_∞ -algebras or, equivalently, an explicit diagonal for the operad of cellular chains, over the integers, of the Stasheff associahedron. These construction were in fact already indicated in [9]; we will try to give a more satisfactory presentation. We also prove that there does not exist a co-associative diagonal.

Table of content: 1. Introduction – page 1

2. Categorial properties of diagonals and tensor products – page 3
3. Calculus of oriented cell complexes of K_n and W_n – page 4
4. The chain maps p and q – page 7
5. The Saneblidze-Umble diagonal – page 11
6. Non-existence of a co-associative diagonal – page 13
7. Remaining proofs – page 16

1. Introduction.

In this paper we study tensor products of A_∞ -algebras. More precisely, given two A_∞ -algebras $A = (V, \partial^V, \mu_2^V, \mu_3^V, \dots)$ and $B = (W, \partial^W, \mu_2^W, \mu_3^W, \dots)$, we will be looking for a functorial definition of an A_∞ -structure $A \odot B$ that would extend the standard (non-associative) dg-algebra structure on the tensor product $A \otimes B$. This means that the A_∞ -algebra $A \odot B$ will be of the form $(V \otimes W, \partial, \mu_2, \mu_3, \dots)$, where ∂ is the usual differential on the tensor product,

$$(1) \quad \partial(v \otimes w) := \partial^V(v) \otimes w + (-1)^{\deg v} v \otimes \partial^W(w)$$

and the bilinear product μ_2 is given by another standard formula

$$(2) \quad \mu_2(v' \otimes w', v'' \otimes w'') := (-1)^{\deg v'' \deg w'} \mu^V(v', v'') \otimes \mu^W(w', w''),$$

where $v, v', v'' \in V$ and $w, w', w'' \in W$.

A “coordinate-free” formulation of the problem is the following. Let \mathcal{A} be the non- Σ operad describing A_∞ -algebras (see [8, page 45]), that is, the minimal model of the non- Σ operad \mathcal{Ass} for associative algebras. The above product is equivalent to a morphism of dg-operads

*Supported by the grant GA ČR 1019203.

†Supported by the Israel Academy of Sciences.

(a *diagonal*) $\Delta : \underline{\mathcal{A}} \rightarrow \underline{\mathcal{A}} \otimes \underline{\mathcal{A}}$ such that Δ induces the usual diagonal $\Delta_{\underline{\mathcal{A}}_{\mathcal{S}}}$ on the non- Σ operad $\underline{\mathcal{A}}_{\mathcal{S}} = H_*(\underline{\mathcal{A}})$.

The existence of such a diagonal is not surprising and follows from properties of minimal models for operads, see [8, Proposition 3.136]. On the other hand, there is no way to control the co-associativity of diagonals constructed using this general argument and we will see below, in Theorem 13, that there, surprisingly enough, *does not exist* a co-associative diagonal.

For practical purposes, such as applications in open string theory [2], one needs a tensor product (and therefore also a diagonal) given by an explicit formula. Such an explicit diagonal was constructed by Umble and Saneblidze in [9]. Our work was in fact motivated by our unsuccessful attempts to understand their paper. We will denote this diagonal by Δ^{su} and call it the *SU-diagonal*. In this article we recall the definition of this diagonal and give a conceptual explanation why it is well-defined. The operad $\underline{\mathcal{A}}$ can be identified with the operad of the cellular chain complexes of the non- Σ operad of associahedra, $\underline{\mathcal{A}} \cong \mathcal{C}_*(\mathcal{K})$ (see [8, page 45]), therefore, the required diagonal is given by a family of chain maps

$$\Delta_{K_n} : \mathcal{C}_*(K_n) \rightarrow \mathcal{C}_*(K_n) \otimes \mathcal{C}_*(K_n), \quad n \geq 1,$$

commuting with the induced operad structures and such that $H_*(\Delta_{K_n}) = \Delta_{\underline{\mathcal{A}}_{\mathcal{S}}}$.

The cells of the associahedra are not conducive to the definition of a diagonal. There is, however, a cubical decomposition of the associahedra provided by the *W-construction* of Boardman and Vogt [1], which is a homotopically equivalent non- Σ operad $\underline{\mathcal{W}} = \{W_n\}_{n \geq 1}$, for which there is a canonical diagonal

$$\Delta_{W_n} : \mathcal{C}_*(W_n) \longrightarrow \mathcal{C}_*(W_n) \otimes \mathcal{C}_*(W_n), \quad n \geq 1,$$

induced by the cubical structure (see (14)). A suitable diagonal on the associahedra can be then obtained by transferring Δ_{W_n} from $\underline{\mathcal{W}}$ to \mathcal{K} . More precisely, let

$$(3) \quad \mathcal{C}_*(W_n) \xrightarrow{p_n} \mathcal{C}_*(K_n) \xrightarrow{q_n} \mathcal{C}_*(W_n), \quad n \geq 1,$$

be arbitrary operadic maps such that $H_*(p_n)$ and $H(q_n)$ are identity endomorphisms of $\underline{\mathcal{A}}_{\mathcal{S}}(n)$, via the canonical identifications

$$H_*(\mathcal{C}_*(W_n)) \cong \underline{\mathcal{A}}_{\mathcal{S}}(n) \cong H_*(\mathcal{C}_*(K_n)), \quad n \geq 1.$$

Then the formula

$$(4) \quad \Delta_{K_n} := (p_n \otimes p_n) \circ \Delta_{W_n} \circ q_n$$

clearly defines a diagonal. In fact, it can be proved that the operadic maps $p = \{p_n\}_{n \geq 1}$ and $q = \{q_n\}_{n \geq 1}$ with the above properties are homotopy inverses, but we will not need this statement.

It remains to find maps in (3). While there is an obvious and simple definition of q_n , finding a suitable formula for p_n is much less obvious. We give an explicit and very natural definition inspired by a formula in [9].

We will see that the operad of cellular chains $\mathcal{C}_*(W)$ can be described in terms of metric trees. Similar *cellular W-constructions* on a given dg-operad were considered by Kontsevich and Soibelman in [4]. In this terminology, the chain maps p and q are explicit homotopy equivalences, defined over the integers, between the chain W -construction on the operad $\underline{\mathcal{A}ss}$ and the minimal model \mathcal{A} of $\underline{\mathcal{A}ss}$, which give rise to explicit equivalences of the categories of algebras over these dg-operads.

2. Categorial properties of diagonals and tensor products.

Recall [5] that there are two notions of morphisms of A_∞ -algebras. A *strict morphism* of A_∞ -algebras $(X, \partial, \mu_2, \mu_3, \dots)$ and $(Y, \partial, \nu_2, \nu_3, \dots)$ is a linear map $f : X \rightarrow Y$ that commutes with all structure operations. A weaker notion is that of a strongly homotopy (sh) morphism, given by a sequence of maps $f_n : X^{\otimes n} \rightarrow Y$, $n \geq 1$, satisfying rather complicated set of axioms (see, for example, [5, 7]). Such a map is invertible if and only if $f_1 : X \rightarrow Y$ is an isomorphism. We will denote by $\mathbf{str}\mathcal{A}_\infty$ the category of A_∞ -algebras and their strict morphisms, and $\mathbf{sh}\mathcal{A}_\infty$ the category of A_∞ -algebras and their sh morphisms.

As proved in [8, Proposition 3.136], any two diagonals $\Delta', \Delta'' : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ are homotopic as maps of operads. Let \odot' (resp. \odot'') denotes the tensor product induced by \odot' (resp. \odot''). Although $A \odot' B$ and $A \odot'' B$ are, in general, not strictly isomorphic, the homotopy between Δ' and Δ'' can be shown to induce a strongly homotopy isomorphism between $A \odot' B$ and $A \odot'' B$. Therefore we obtain the following uniqueness:

Proposition 1. *For any two A_∞ -algebras A, B , the A_∞ -algebras $A \odot' B$ and $A \odot'' B$ are isomorphic in $\mathbf{sh}\mathcal{A}_\infty$.*

We will prove in Theorem 13 that there are no co-associative diagonals. This means that in general

$$A \odot (B \odot C) \not\cong (A \odot B) \odot C$$

in the ‘strict’ category $\mathbf{str}\mathcal{A}_\infty$. On the other hand, as argued in [8, Proposition 3.136], each diagonal Δ is *homotopy associative* in the sense that the maps $(\Delta \otimes \mathbb{1})\Delta$ and $(\mathbb{1} \otimes \Delta)\Delta$ are homotopic maps of operads, from which we infer:

Proposition 2. *For any three A_∞ -algebras A, B and C ,*

$$A \odot (B \odot C) \cong (A \odot B) \odot C$$

in the ‘weak’ category $\mathbf{sh}\mathcal{A}_\infty$.

By the same argument, one can also prove

Proposition 3. *For any two A_∞ -algebras A and B ,*

$$A \odot B \cong B \odot A$$

in $\mathbf{sh}\mathcal{A}_\infty$.

This naturally rises the question whether \mathbf{shA}_∞ with a product \odot based on an appropriate diagonal is a (possibly symmetric) monoidal category. Even to formulate this question precisely, one more step should be completed.

While it is clear that \odot is a functor $\mathbf{strA}_\infty \times \mathbf{strA}_\infty \rightarrow \mathbf{strA}_\infty$, to make it a functor $\mathbf{shA}_\infty \times \mathbf{shA}_\infty \rightarrow \mathbf{shA}_\infty$, one should define, for two sh morphisms $f : A' \rightarrow A''$ and $g : B' \rightarrow B''$, a ‘product’ $f \odot g : A' \odot A'' \rightarrow B' \odot B''$. One should then consider a functorial ‘associator’ $\Phi_{A,B,C} : A \odot (B \odot C) \rightarrow (A \odot B) \odot C$ and a ‘symmetry’ $\sigma_{A,B} : A \odot B \rightarrow B \odot A$.

The above objects exist by general nonsense, but it is not clear whether they fulfill the axioms of a (symmetric) monoidal category (the pentagon and the hexagons), although it is quite possible that for some special choices of the above data these axioms are satisfied. On a more abstract level, the ‘full’ functorial monoidal product $A, B \mapsto A \odot B$, $f, g \mapsto f \odot g$ in \mathbf{shA}_∞ means to construct a ‘diagonal’ in the minimal model of the two-colored operad $\underline{Ass}_{\bullet \rightarrow \bullet}$ describing homomorphisms of associative algebras, satisfying some additional properties which do not follow from a general nonsense.

3. Calculus of oriented cell complexes of K_n and W_n .

All operads \mathcal{P} considered in this paper are such that $\mathcal{P}(0)$ is trivial and that $\mathcal{P}(1)$ is isomorphic to the ground field. The category of operads with this property is equivalent to the category of *pseudo-operads* \mathcal{P} such that $\mathcal{P}(0) = \mathcal{P}(1) = 0$, the equivalence being given by forgetting the $n = 1$ piece. This, roughly speaking, means that we may ignore operadic units, see [6, Observation 1.2] for details. *Therefore, for the rest of this paper, an operad means a pseudo-operad with $\mathcal{P}(0) = \mathcal{P}(1) = 0$.*

First, we establish some notation. Let $\underline{\mathcal{K}} = \{K_n\}_{n \geq 2}$ be the non- Σ operad of associahedra. The topological cell complex K_n can be realized as a convex polytope in \mathbf{R}^{n-2} , with k -cells labeled by the planar rooted trees with n leaves and $n - k - 2$ internal edges, or equivalently by $(n - k - 2)$ -fold bracketings of n elements, see [8, II.1.6]. For example, 0-cells correspond to binary trees with n leaves, or equivalently, full bracketings of n elements. All our constructions will be expressed in terms of rooted planar trees although there is clearly an underlying geometric meaning based on the polytope realization of K_n . Boardman and Vogt have defined in [1] a cubical subdivision of the cells of K_n , for $n \geq 2$, giving rise to a cubical cell complex known as the W -construction, W_n . See Figure 6 of [8, Section II.2.8] for W_4 represented as a cubical subdivision of K_4 .

The cells of W_n are in one-to-one correspondence with “metric n -trees,” that is, planar rooted trees with n leaves and with internal edges labeled either “metric” or “non-metric.” The metric n -trees with k metric edges label the topological k -cells of W_n . A cubical cell is called *an interior cell* if the labeling tree has only metric edges. In the geometric realization the interior cells are in the interior of the convex polytope.

In order to define the boundary operators on the complexes $\mathcal{C}_*(\underline{\mathcal{K}}) := \{\mathcal{C}_*(K_n)\}_{n \geq 2}$ and $\mathcal{C}_*(\underline{W}) := \{\mathcal{C}_*(W_n)\}_{n \geq 2}$ (non- Σ operads in the category of chain complexes), we have to introduce an orientation on the cells. Let T be a planar rooted tree with internal edges labeled e_1, \dots, e_m . Two orderings e_{i_1}, \dots, e_{i_m} and e_{j_1}, \dots, e_{j_m} will be called *equivalent* if they are related by an even permutation. The equivalence class corresponding to an ordering

e_{i_1}, \dots, e_{i_m} will be called an orientation and denoted $e_{i_1} \wedge \dots \wedge e_{i_m}$.

Definition 4. An oriented k -cell in K_n is a pair (T, ω) where T is a planar rooted tree with n leaves and $n - k - 2$ internal edges and ω is an orientation. Let $\mathcal{C}_k(K_n)$ be the vector space spanned by the oriented k -cells in K_n modulo the relation $(T, \omega) = -(T, \omega')$ where ω and ω' are the two distinct orientations.

An oriented metric k -cell in W_n is a pair (T, ω) where T is a metric tree with n leaves and k metric edges and ω is an orientation of the metric edges. Let $\mathcal{C}_k(W_n)$ be the vector space spanned by the oriented k -cells in W_n modulo the relation $(T, \omega) = -(T, \omega')$ where ω and ω' are the two distinct orientations.

The operad composition law

$$\circ_i : \mathcal{C}_k(K_r) \otimes \mathcal{C}_l(K_s) \longrightarrow \mathcal{C}_{k+l}(K_{r+s-1})$$

is defined on the basis elements by

$$(5) \quad (T, \omega) \circ_i (T', \omega') := (-1)^{rl+i(s+1)} (T \circ_i T', \omega \wedge \omega' \wedge e),$$

where \circ_i is defined on planar rooted trees in the standard way, grafting the second tree onto the i -th leaf of the first, and $\omega \wedge \omega' \wedge e$ is the concatenation of the two orientations, with the new edge created by grafting labeled e . The operad composition law

$$\circ_i : \mathcal{C}_k(W_r) \otimes \mathcal{C}_l(W_s) \longrightarrow \mathcal{C}_{k+l}(W_{r+s-1})$$

is defined on the basis elements by

$$(6) \quad (T, \omega) \circ_i (T', \omega') := (T \circ_i T', \omega \wedge \omega'),$$

A heuristic explanation of why we don't need any signs in the above display is that the orientation of the cells of \mathcal{W} defined in terms of metric edges is geometric in the sense that the number of metric edges is the same as the dimension of the cell. In the case of $\mathcal{C}_*(\mathcal{W})$ the new edge created by the grafting is non-metric and so does not appear in the ordering of metric vertices.

The boundary operator on $\mathcal{C}_*(K_n)$ is defined by

$$(7) \quad \partial_K(T, e_1 \wedge \dots \wedge e_m) := \sum_{\{T' \mid T'/e' = T\}} (T', e' \wedge e_1 \wedge \dots \wedge e_m),$$

where the sum is over all trees T' with an edge e' , such that when e' is collapsed T' reduces to T . The condition $\partial_K^2 = 0$ follows immediately from the identities

$$(T'', e' \wedge e'' \wedge e_1 \wedge \dots \wedge e_m) = -(T'', e'' \wedge e' \wedge e_1 \wedge \dots \wedge e_m).$$

Next we define the boundary operator on the complex $\mathcal{C}_*(W_n)$. Let T be a metric tree and $e_1 \wedge \dots \wedge e_k$ an orientation:

$$(8) \quad \begin{aligned} \partial_W(T, e_1 \wedge \dots \wedge e_k) := & \\ & \sum_{1 \leq i \leq k} (-1)^i [(T/e_i, e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_k) - (T_i, e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_k)], \end{aligned}$$

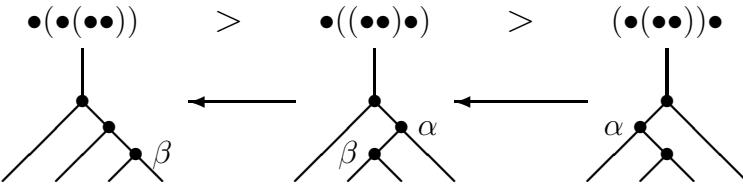


Figure 1: The partial order on the set of binary trees. The first on the right arrow moves the vertex α and the second arrow moves the vertex β .

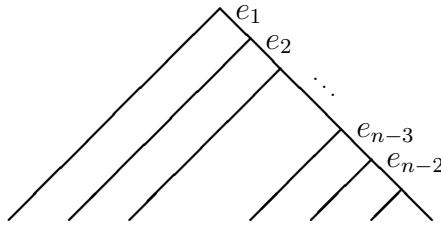


Figure 2: The maximal binary tree $\bar{b}(n)$.

where T_i is the same (unlabeled) tree as T but with the metric edge e_i changed to a non-metric edge. As above, the condition $\partial_W^2 = 0$ follows from the relations for the orientation elements.

In the rest of this section we introduce ‘standard orientations’ for top dimensional cells of W_n and 0-dimensional cells of K_n . There is a partial order relation on rooted planar binary trees given by the associator which moves a vertex to the right and changes the outgoing edge from a right leaning position to a left-leaning position, as shown in Figure 1. The *standard orientation* $\omega_{\bar{b}(n)}$ of the maximal fully metric binary tree $\bar{b}(n)$ (all internal edges leaning to the left) is given by enumerating the internal edges in sequence, starting with e_1 , the edge adjacent to the root, and continuing e_2, \dots, e_{n-2} in sequence going away from the root, see Figure 2.

The standard orientation ω_T of a non-maximal fully metric binary tree T is determined by a sequence of sign changes and relabelings along a path from $\bar{b}(n)$ to T in the associahedron. See Figure 3 for the standard orientations of binary trees with four leaves. To check that this rule gives an unambiguous definition of the orientation, it is sufficient (thanks to Mac Lane’s Coherence Theorem) to verify that the definition is independent of path in the pentagon (expressing coherence of the associator) and in the square (expressing naturality). The verification for the pentagon is given in Figure 3. The verification for the square is straightforward and follows from the functoriality. Therefore each fully metric binary n -tree T together with its standard orientation ω_T determines an element $(T, \omega_T) \in \mathcal{C}_{n-2}(W_n)$.

We also define the *standard orientation* ξ_T of a binary n -tree T representing a 0-cell of $\mathcal{C}_0(K_n)$ inductively as follows. The only binary 2-tree representing a 0-cell of $\mathcal{C}_0(K_2)$ has no internal edges, and its canonical orientation is given by assigning the +1-sign to this tree. The canonical orientation of any binary tree would be then determined by the formula

$$(S, \xi_S) \circ (T, \xi_T) = (S \circ_i T, \xi_{S \circ_i T}),$$

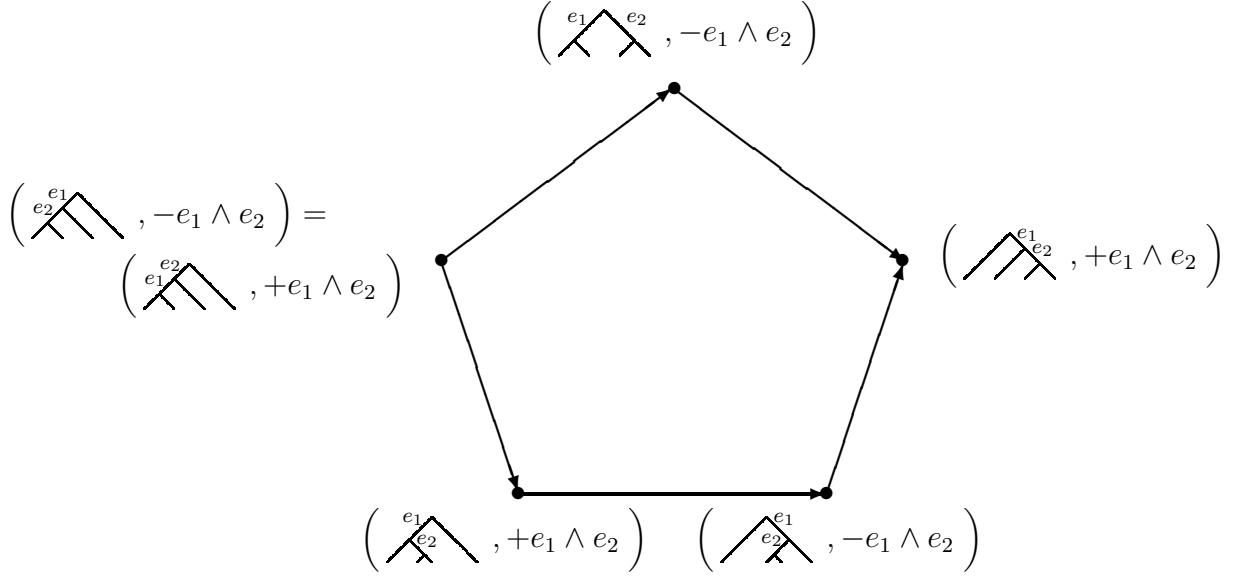


Figure 3: The rule for defining the standard orientation of fully metric trees is illustrated for the pentagon in the figure above. This example also verifies that the definition is independent of the path.

once we checked that there was no ambiguity. This can be done exactly as in the previous paragraph for ω_T . For example, we immediately get the following standard orientations:

$$(\overbrace{\nwarrow}^a, -a) \text{ and } (\nearrow^e, e).$$

We recommend as an exercise to verify that the standard orientation $\xi_{\bar{b}(n)}$ of the maximal binary tree in Figure 3, this time considered as a 0 cell of K_n is, for $n > 2$,

$$(\bar{b}(n), \xi_{\bar{b}(n)}) := (-1)^{(n-2)(n-3)/2} \cdot (\bar{b}(n), e_1 \wedge \cdots \wedge e_{n-2}) \in \mathcal{C}_0(K_n)$$

and that the standard orientation of the minimal binary n -tree $\underline{b}(n)$ with the interior edges (all are right-leaning) enumerated in sequence going away from the root, is given as

$$(\underline{b}(n), \xi_{\underline{b}(n)}) := (-1)^n \cdot (\underline{b}(n), e_1 \wedge \cdots \wedge e_{n-2}) \in \mathcal{C}_0(K_n).$$

4. The chain maps p and q .

The goal of this section is to construct maps $p : \mathcal{C}_*(\mathcal{K}) \rightarrow \mathcal{C}_*(\mathcal{W})$ (Definition 7) and $q : \mathcal{C}_*(\mathcal{W}) \rightarrow \mathcal{C}_*(\mathcal{K})$ (Definition 5) with the properties discussed in Section 1. The proofs that that these maps are indeed chain maps (Proposition 10 and Proposition 6) are postponed to Section 7.

As an operad in the category of vector spaces, $\mathcal{C}_*(\mathcal{K})$ is a free operad generated by the collection with arity n component, a one-dimensional subspace concentrated in degree $n - 2$ spanned by corolla with n leaves, and $\mathcal{C}_*(\mathcal{W})$ is a free operad generated by the collection with arity n component, the vector space with basis the set of purely metric planar rooted

trees with n leaves. Since a operadic map of a free operad is determined by its value on generators, the operadic chain map $\mathcal{C}_*(\underline{\mathcal{K}}) \xrightarrow{q} \mathcal{C}_*(\underline{\mathcal{W}})$ is determined by its value on corollae, and the operadic chain map $\mathcal{C}_*(\underline{\mathcal{W}}) \xrightarrow{p} \mathcal{C}_*(\underline{\mathcal{K}})$ is determined by its value on purely metric trees.

Let $c(n)$ be the corolla with n leaves; since there are no internal edges, we denote the orientation by the symbol 1, and adopt the convention that

$$1 \wedge e_1 \wedge \cdots \wedge e_k := e_1 \wedge \cdots \wedge e_k.$$

Definition 5. Let $mBin(n)$ be the set of $n - 2$ cells of W_n corresponding to the fully metric planar rooted binary trees with n leaves and standard orientation. Then $q(c(n), 1)$ is defined as a sum over $mBin(n)$:

$$(9) \quad q(c(n), 1) := \sum_{T \in mBin(n)} (T, \omega_T).$$

The operadic extension of q to the free operad $\mathcal{C}(\underline{\mathcal{K}})$, which map will also be denoted q , defines a morphism of operads in the category of graded vector spaces.

Proposition 6. The morphism q described in Definition 5 commutes with the boundary operators,

$$(10) \quad q(\partial_K(c(n), 1)) = \partial_W q(c(n), 1),$$

and therefore is a morphism of operads in the category of chain complexes.

The proof of Proposition 6 is postponed to Section 7. The operad chain map $p : \mathcal{C}_*(\underline{\mathcal{W}}) \rightarrow \mathcal{C}_*(\underline{\mathcal{K}})$ is determined by its value on fully metric trees. Before giving the precise definition, we will give a conceptual description. As a topological cell complex, the associahedron can be realized as a convex polytope $K_n \subset \mathbf{R}^{n-2}$. The cubical cell complex W_n is a decomposition of the associahedral k -cells into k -cubes. The interior k -cell of W_n labeled by a purely metric tree T with k edges is transverse to the $n - 2 - k$ cell of K_n labeled by the same tree. Let T_{min} be the binary tree labeling the minimal vertex of this transverse cell in K_n .

The image $p(T)$ is defined as the sum with appropriate signs of all the k -cells in $\mathcal{C}_k(K_n)$ all of whose vertices are labeled by binary trees less than or equal to T_{min} relative to the partial order on binary trees.

The tree T_{min} is created by “filling-in” the non-binary vertices of T . A vertex in T with r input edges, $r > 2$, is replaced in T_{min} by the minimal binary tree with r leaves, which introduces $r - 2$ new right-leaning edges. When this procedure is carried out at all the non-binary vertices of T , it adds $n - 2 - k$ new edges, all of which are right-leaning. See Figure 4 for an example of this procedure. In exactly the same way, one defines T_{max} as the binary tree obtained from T by filling-in the non-binary vertices by left-leaning edges.

In order for a binary tree S to be the maximal vertex of a k -cell in K_n , it must contain at least k left-leaning edges, since an associativity move applied to a binary tree replaces a right-leaning edge with a left-leaning edge (see Figure 1) and the tree labeling the maximal vertex of k -cell is the output of at least k distinct associativity moves, corresponding to the k one-cells of the associahedron which meet at the given vertex.

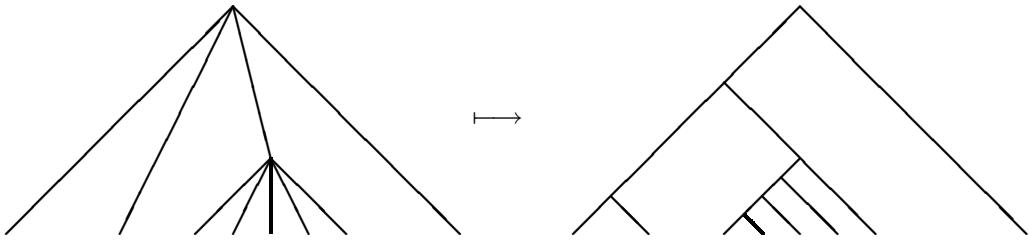


Figure 4: An example of the filling-in procedure passing from a fully metric tree T to the binary tree T_{min} .

If T is an interior k -cell in W_n , then T_{min} cannot have more than k left-leaning edges, since the new edges in T_{min} are all right-leaning. Since the number of left leaning edges in a binary tree is a non-decreasing function relative to the partial order, if T_{min} has less than k left-leaning edges, there are no k -cells less than T_{min} and we put $p(T) = 0$.

Given $(T, e_1 \wedge \cdots \wedge e_k) \in \mathcal{C}_k(W_n)$, such that T_{min} has k left-leaning edges, then each edge e_i corresponds to an edge in T_{min} which we also denote e_i . Choose any labeling $f_1 \wedge \cdots \wedge f_{n-k-2}$ of the new edges, and let $\xi_{T_{min}}$ be the standard orientation of T_{min} considered as the label for a 0-cell of K_n ,

$$\xi_{T_{min}} = \eta \cdot e_1 \wedge \cdots \wedge e_k \wedge f_1 \wedge \cdots \wedge f_{n-k-2}, \quad \eta \in \{-1, +1\},$$

then leading term of $p(T, e_1 \wedge \cdots \wedge e_k)$ will be

$$(11) \quad \begin{aligned} (T_{min}/\{e_1, \dots, e_k\}, e_1 \wedge \cdots \wedge e_k \lrcorner \xi_{T_{min}}) &= \\ &= \eta \cdot (-1)^{k(k-1)/2} (T_{min}/\{e_1, \dots, e_k\}, f_1 \wedge \cdots \wedge f_{n-k-2}). \end{aligned}$$

In the above display, \lrcorner is the contraction relative to the pairing $\langle e_i, e_j \rangle := \delta_{ij}^2$.

For any binary tree $S < T_{min}$ with k left-leaning edges, we will describe a method (analogous to the definition of the standard orientation) of assigning in a unique way a labeling of the left-leaning edges by the labels e_1, \dots, e_k . First, we describe a rule which determines the labeling of the left-leaning edges in a tree given the labeling of the left-leaning edges in an adjacent tree (related by one associativity). Consider a binary tree with labels only on the left-leaning edges, adjacent trees are related by replacing configuration of two edges \triangleright by the configuration \triangleleft (going from the greater tree to the lesser tree). If both edges are internal, the rule is simply to use the same label for the left-leaning edge in both configurations. The ambiguity of the path connecting two trees resolves into a sequence of pentagons and squares and the validity of the definition is checked by considering these two figures. On the other hand, if the lower edge is a leaf, the new configuration has a right-leaning edge in place of a left-leaning edge and the resulting binary tree has less than k left-leaning edges so that there are no k -cells less than it and the contribution to $p(T)$ is zero. For example, in Figure 1, one of the associativity moves preserves the number of left-leaning edges and the other changes the number by one.

We can now give the full definition of p :

Definition 7. Define a function on oriented fully metric trees $(T, e_1 \wedge \dots \wedge e_k)$ by

$$(12) \quad p(T, e_1 \wedge \dots \wedge e_k) := \sum(S/\{e_1, \dots, e_k\}, e_1 \wedge \dots \wedge e_k \sqcup \xi),$$

where the sum is over binary trees S less than or equal to T_{min} with k left-leaning edges labeled e_1, \dots, e_k according to the procedure described above, ξ is the standard orientation of the binary tree S and \sqcup is the same contraction as in (11).

The function p has a unique extension to a morphism (denoted also by the same symbol) $p : \mathcal{C}_*(W_*) \rightarrow \mathcal{C}_*(K_*)$ of operads in the category of graded vector spaces.

Exercise 8. Verify that

$$p(\overline{b}(n), \omega_{\overline{b}(n)}) = (c(n), 1) \text{ and } p(c(n), 1) = (\underline{b}(n), \xi_{\underline{b}}).$$

Note that the first equation involves $(n-2)$ -cells and the second involves 0-cells. Observe also that, modulo orientations, (12) is the sum of all trees U with $n-k$ interior edges such that $U_{max} \leq T_{min}$.

Example 9. Let us describe explicitly the map $p : \mathcal{C}_*(W_n) \rightarrow \mathcal{C}_*(K_n)$ for some small n . For $n=1$ and 2, p is given by

$$p(|, 1) := (|, 1) \text{ and } p(\nwarrow, 1) := (\nwarrow, 1).$$

For $n=3$,

$$p(\nearrow \nwarrow, 1) := \left(\begin{smallmatrix} a & \\ \nearrow & \nwarrow \\ & -a \end{smallmatrix} \right), \quad p(\nearrow \nearrow^e, e) := \left(\begin{smallmatrix} & \\ \nearrow & \nearrow \\ & e \end{smallmatrix} \right), \quad p(\nearrow \nearrow^e, e) := 0,$$

where e denotes a metric edge of W_3 . Finally, for $n=4$,

$$\begin{aligned} p(\nearrow \nearrow \nwarrow, 1) &:= \left(\begin{smallmatrix} a & \\ \nearrow & \nearrow \\ & \nwarrow \\ & -a \end{smallmatrix} \right), \\ p(\nearrow \nearrow^e, e) &:= \left(\begin{smallmatrix} & \\ \nearrow & \nearrow \\ & e \end{smallmatrix} \right) + \left(\begin{smallmatrix} a & \\ \nearrow & \nearrow \\ & e \end{smallmatrix} \right), \\ p(\nearrow \nearrow^e, e) &:= \left(\begin{smallmatrix} a & \\ \nearrow & \nearrow \\ & e \end{smallmatrix} \right), \\ p(\nearrow \nearrow^e, e) &:= \left(\begin{smallmatrix} a & \\ \nearrow & \nearrow \\ & e \end{smallmatrix} \right) \text{ and} \\ p(\nearrow \nearrow^e \nearrow^f, e \wedge f) &:= (\nearrow \nearrow \nwarrow, 1), \end{aligned}$$

where e and f are metric edges of W_4 . The above equations can be written in a more condensed form as

$$\begin{aligned} p(|) &= |, \quad p(\nearrow) = \nearrow, \quad p(\nwarrow) = \nwarrow, \quad p(\nwarrow) = \nwarrow, \quad p(\nwarrow) = 0, \\ p(\nwarrow) &= \nwarrow, \quad p(\nwarrow) = \nwarrow + \nwarrow, \quad p(\nwarrow) = \nwarrow, \quad p(\nwarrow) = -\nwarrow \text{ and } p(\nwarrow) = \nwarrow, \end{aligned}$$

with the convention that binary trees are endowed with their canonical orientations, corollas are oriented with the $+$ sign and trees T with one binary and one ternary vertex are oriented as (T, e) , where e denotes the unique interior edge of T .

Let us close this section by the following proposition whose proof is postponed to Section 7.

Proposition 10. *Let (T, ω_T) be an oriented fully metric tree, then*

$$(13) \quad p(\partial_W(T, \omega_T)) = \partial_K p(T, \omega_T).$$

Since p is a operad morphism, this implies that p commutes with the differential on $\mathcal{C}_(W_*)$ and therefore is a morphism of operads in the category of chain complexes.*

5. The Saneblidze-Umble diagonal.

In this section we define the SU-diagonal [9]. Let us start with a definition of the cubical diagonal Δ_W adapted from [10, Section 2]:

$$(14) \quad \begin{aligned} \Delta_W(T, e_1 \wedge \cdots \wedge e_k) := \\ := \sum_{L,R} (-1)^{\rho_{L,R}} (T/e_L, e_1 \wedge \cdots \hat{e}_{i_1} \cdots \hat{e}_{i_l} \cdots \wedge e_k) \otimes (T_R, e_1 \wedge \cdots \hat{e}_{j_1} \cdots \hat{e}_{j_r} \cdots \wedge e_k), \end{aligned}$$

where the summation runs over all disjoint decompositions $L \sqcup R = \{i_1, \dots, i_l\} \sqcup \{j_1, \dots, j_r\}$ of $\{1, \dots, k\}$ into ordered subsets, T/e_L is the tree obtained from T by contracting edges $\{e_i; i \in L\}$, T_R is the tree obtained by changing the metric edges $\{e_j; j \in R\}$ to non-metric ones, and $\rho_{L,R}$ is the number of couples $i \in L, j \in R$ such that $i < j$. We leave as an exercise to prove:

Proposition 11. *The diagonal (14) is co-associative and commutes with the \circ_i -operations introduced in (6), therefore the W -construction (\mathcal{W}, Δ_W) is a Hopf non- Σ operad.*

The SU-diagonal is then defined by formula (4), that is

$$(15) \quad \Delta^{\text{su}} := (p \otimes p) \circ \Delta_W \circ q.$$

Exercise 12. Derive from definition that, in the shorthand of Example 9,

$$(16) \quad \begin{aligned} \Delta^{\text{su}}(\text{ } \diagdown \text{ }) &= \text{ } \diagdown \text{ } \otimes \text{ } \diagdown \text{ }, \\ \Delta^{\text{su}}(\text{ } \diagup \text{ }) &= \text{ } \diagup \text{ } \otimes \text{ } \diagup \text{ } + \text{ } \diagup \text{ } \otimes \text{ } \diagup \text{ }, \\ \Delta^{\text{su}}(\text{ } \diagup \text{ } \diagdown \text{ }) &= \text{ } \diagup \text{ } \otimes \text{ } \diagdown \text{ } + \text{ } \diagup \text{ } \otimes \text{ } \diagup \text{ } + \text{ } \diagdown \text{ } \otimes \text{ } \diagup \text{ } + \text{ } \diagdown \text{ } \otimes \text{ } \diagdown \text{ } - \text{ } \diagup \text{ } \otimes \text{ } \diagdown \text{ } + \text{ } \diagup \text{ } \otimes \text{ } \diagup \text{ }, \text{ etc.} \end{aligned}$$

Prove also that $\Delta^{\text{su}}(c(n), 1)$ always contains the terms

$$(\underline{b}(n), \xi_{\underline{b}(n)}) \otimes (c(n), 1) \quad \text{and} \quad (c(n), 1) \otimes (\bar{b}(n), \xi_{\bar{b}(n)}).$$

Let us analyze formula (15) applied to $(c(n), 1)$. The map q_n applied to the oriented corolla $(c(n), 1) \in \mathcal{C}_{n-2}(K_n)$ is, by definition, the sum of all fully metric binary trees with standard orientations. The diagonal Δ_{W_n} acts on such a tree (S, ω_S) as follows. Divide interior edges of S into two disjoint groups, $\{f_1, \dots, f_s\}$, $\{e_1, \dots, e_t\}$, $t + s = n - 2$, and let

$$\omega_S = \eta \cdot e_1 \wedge \cdots \wedge e_t \wedge f_1 \wedge \cdots \wedge f_s,$$

with some $\eta \in \{-1, 1\}$, be the standard orientation.

Then $\Delta_{W_n}(S)$ contains the term $(S_L, e_1 \wedge \dots \wedge e_t) \otimes (S_R, f_1 \wedge \dots \wedge f_s)$, where $S_L = S/\{f_1, \dots, f_s\}$ and S_R is obtained by replacing edges $\{e_1, \dots, e_t\}$ of S by non-metric ones. We must then evaluate

$$(17) \quad p_n(S_L, e_1 \wedge \dots \wedge e_t) \otimes p_n(S_R, f_1 \wedge \dots \wedge f_s).$$

One can also describe the pair S_L, S_R as follows: S_R is the same binary tree as S , but with only a subset of the edges retaining the metric label, S_L is the fully-metric tree formed from S by collapsing the same subset of edges.

Let us pause a little and observe that the expression in (17) is nonzero only for trees S of a very special form. Since the value $p(U, \omega)$ is, for a binary fully metric tree U , nonzero only when U is maximal, $p_n(S_R, f_1 \wedge \dots \wedge f_s)$ is nontrivial only when S_R is build from maximal binary fully metric trees, using the \circ -operation t -times. Similarly, as we saw in Section 4, $p_n(S_L, e_1 \wedge \dots \wedge e_t)$ is nonzero if and only if $(S_L)_{min}$ has exactly t left leaning edges.

A moment's reflection convinces us that the above two conditions are satisfied if and only if S_R is build up from $t+1$ fully metric maximal binary trees, using t times \circ_i -operations with $i \geq 2$ (that is, \circ_1 is forbidden). Clearly $p_n(S_R, f_1 \wedge \dots \wedge f_s)$ is then an n -tree created from $t+1$ corollas using \circ_i with $i \geq 2$. Let M_n^t denote the set of such n -trees and $M_n := M_n^0 \sqcup \dots \sqcup M_n^{n-2}$. A more formal definition is that $T \in M_n$ if and only if $T = c(k) \circ_i S$ for some $S \in M_l$, where $k+l = n-1$ and $i \geq 2$. For example,

$$M_1 = \{|\}, \quad M_2 = \{\nwarrow\}, \quad M_3 = \{\nwarrow, \nearrow\}, \quad M_4 = \{\nwarrow, \nearrow, \nwarrow, \nearrow, \nwarrow\}, \quad \text{etc.}$$

We recommend to prove as an exercise that M_n is the set of all n trees T whose number of interior edges is the same as the number of left leaning edges of T_{min} .

Let us reverse the process and start with an oriented n -tree $(T, \xi) \in \mathcal{C}_s(K_n)$ such that $T \in M_n^t$ and $\xi = e_1 \wedge \dots \wedge e_t$. Let \tilde{T} be the tree obtained from T by filling all non-binary vertices by left-leaning metric edges. Let us denote these newly created metric edges f_1, \dots, f_s . Observe that

$$p_n(\tilde{T}_R, f_1 \wedge \dots \wedge f_s) = \epsilon \cdot (T, \xi)$$

for some $\epsilon \in \{-1, +1\}$. Define $\eta_T \in \{-1, +1\}$ by demanding $\eta_T \cdot \epsilon \cdot e_1 \wedge \dots \wedge e_t \wedge f_1 \wedge \dots \wedge f_s$ to be the standard orientation of \tilde{T} . It is not hard to prove that η_T indeed depends only on T and not on the choices of the labels $e_1, \dots, e_t, f_1, \dots, f_s$, as suggested by the notation. For example, $\eta_T = 1$ for all trees from $T \in M_n$ with $n \leq 1$ except $T = \nwarrow = b(4)$ for which $\eta_T = -1$. More generally, $\eta_{b(n)} = (-1)^{(n-2)(n-3)/2}$.

Observe finally that $\tilde{T}_L = T$. Equation (15) can then be rewritten as

$$(18) \quad \Delta^{\text{su}}(c(n), 1) = \sum_{\substack{T \in M_n^t \\ 0 \leq t \leq n-1}} \eta_T \cdot p_n(T, e_1 \wedge \dots \wedge e_t) \otimes (T, e_1 \wedge \dots \wedge e_t).$$

Let us notice that the above display contains the symbol $(T, e_1 \wedge \dots \wedge e_t)$ twice. The first occurrence of this symbol denotes a cell of $\mathcal{C}_t(W_n)$, the second occurrence a cell of $\mathcal{C}_s(K_n)$. The sign η_T then accounts for the difference between these two interpretations of the same symbol.

We already observed in Example 8 that, modulo orientations, $p_n(T, e_1 \wedge \dots \wedge e_t)$ in (18) is the sum of all n -trees U with s interior edges such that $U_{max} \leq T_{min}$. This leads to the following formula for the SU-diagonal whose spirit is closer to [9]:

$$(19) \quad \Delta^{su}(c(n), 1) = \sum \vartheta \cdot (U, \omega_U) \otimes (T, \omega_T),$$

where, as usual, $c(n)$ is the n -corolla representing the top dimensional cell of K_n , the summation is taken over all (U, ω_U) , (T, ω_T) with $U_{max} \leq T_{min}$ and $\dim(S, \omega_S) + \dim(T, \omega_T) = n$, and ϑ is a sign which can be picked up by comparing this formula to (18).

6. Non-existence of a co-associative diagonal.

As we already indicated, the SU-diagonal is not co-associative, that is,

$$(20) \quad (\Delta^{su} \otimes \mathbb{1})\Delta^{su} \neq (\mathbb{1} \otimes \Delta^{su})\Delta^{su}.$$

While still

$$(\Delta^{su} \otimes \mathbb{1})\Delta^{su}(\curlywedge) = (\mathbb{1} \otimes \Delta^{su})\Delta^{su}(\curlywedge) \text{ and } (\Delta^{su} \otimes \mathbb{1})\Delta^{su}(\curlywedge) = (\mathbb{1} \otimes \Delta^{su})\Delta^{su}(\curlywedge),$$

the co-associativity breaks already for \curlywedge , explicitly:

$$(21) \quad (\Delta^{su} \otimes \mathbb{1})\Delta^{su}(\curlywedge) - (\mathbb{1} \otimes \Delta^{su})\Delta^{su}(\curlywedge) = \partial(\curlywedge \otimes \curlywedge \otimes \curlywedge)$$

The SU diagonal is also not co-commutative. This means that

$$T(\Delta^{su}) \neq \Delta^{su},$$

where $T : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ is the ‘flip.’ More explicitly, while $T(\Delta^{su})(\curlywedge) = \Delta^{su}(\curlywedge)$,

$$\Delta^{su}(\curlywedge) - T(\Delta^{su})(\curlywedge) = \partial(\curlywedge \otimes \curlywedge).$$

In the rest of this section we show that the non-coassociativity of Δ^{su} is not due to bad choices in the definition, but follows from a deeper principle, namely:

Theorem 13. *The operad \mathcal{A} does not admit a co-associative diagonal. Therefore the operad \mathcal{A} for A_∞ -algebras is not a Hopf operad in the sense of [3].*

Proof. The proof is boring and the reader is warmly encouraged to skip it. The idea is to try to construct inductively a co-associative diagonal Δ and observe that at a certain stage there is a non-trivial co-associativity constraint. Let us start with the construction. For \curlywedge we are forced to take

$$\Delta(\curlywedge) := \curlywedge \otimes \curlywedge.$$

The most general form of $\Delta(\curlywedge)$ is

$$\Delta(\curlywedge) = (a\curlywedge + b\curlywedge) \otimes \curlywedge + \curlywedge \otimes (c\curlywedge + d\curlywedge),$$

with some $a, b, c, d \in \mathbf{k}$. The compatibility with the differential ∂ of $\underline{\mathcal{A}}$ means that

$$\partial\Delta(\mathbb{1}) = (a\mathbb{1} + b\mathbb{1}) \otimes (\mathbb{1} - \mathbb{1}) + (\mathbb{1} - \mathbb{1}) \otimes (c\mathbb{1} + d\mathbb{1})$$

must be the same as

$$\Delta(\partial\mathbb{1}) = \Delta(\mathbb{1} - \mathbb{1}) = \mathbb{1} \otimes \mathbb{1} - \mathbb{1} \otimes \mathbb{1}.$$

This is clearly equivalent to

$$a + c = 1, \quad b + d = 1, \quad a = d \text{ and } b = c.$$

It can be equally easily verified that the co-associativity

$$(\Delta \otimes \mathbb{1})\Delta(\mathbb{1}) = (\mathbb{1} \otimes \Delta)\Delta(\mathbb{1})$$

is equivalent to

$$a = a^2, \quad b = b^2, \quad c = c^2, \quad d = d^2, \quad ab = 0 \text{ and } cd = 0.$$

We conclude that the only two co-associative solutions are either $(a, b, c, d) = (1, 0, 0, 1)$ or $(a, b, c, d) = (0, 1, 1, 0)$, that is either

$$(22) \quad \Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1}.$$

or

$$(23) \quad \Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1}.$$

Let us assume solution (22) which coincides with the SU-diagonal (compare (16)) – solution (23) is just the flip $T(\Delta^{\text{su}}(\mathbb{1}))$ and this case can be discussed by flipping all the steps below. We will be looking for Δ of the form $\Delta = \Delta^{\text{su}} + \delta$ with some perturbation $\delta : \underline{\mathcal{A}} \rightarrow \underline{\mathcal{A}} \otimes \underline{\mathcal{A}}$ satisfying, of course, $\delta(\mathbb{1}) = \delta(\mathbb{1}) = 0$. Since we know that Δ^{su} is a chain map, δ must be a chain map as well.

Observe that $\delta(\mathbb{1})$ depends on 35 parameters. Therefore the co-associativity of Δ and the chain condition on δ is expressed by a system of linear equations in 35 variables! We are going to show that this system has no solution. This might be a formidable task, but we will simplify it by making some wise guesses. Let us write

$$(24) \quad \delta(\mathbb{1}) = A \otimes \mathbb{1} + \sum_i J_{(1)}^i \otimes J_{(2)}^i + \mathbb{1} \otimes B,$$

where $A, B \in \underline{\mathcal{A}}_0(4)$ and $J_{(1)}^i \otimes J_{(2)}^i \in \underline{\mathcal{A}}_1(4) \otimes \underline{\mathcal{A}}_1(4)$. Let us also denote

$$LHS := [(\delta \otimes \mathbb{1})\delta + (\Delta^{\text{su}} \otimes \mathbb{1})\delta + (\delta \otimes \mathbb{1})\Delta^{\text{su}} + (\Delta^{\text{su}} \otimes \mathbb{1})\Delta^{\text{su}}](\mathbb{1})$$

and

$$RHS := [(\mathbb{1} \otimes \delta)\delta + (\mathbb{1} \otimes \Delta^{\text{su}})\delta + (\mathbb{1} \otimes \delta)\Delta^{\text{su}} + (\mathbb{1} \otimes \Delta^{\text{su}})\Delta^{\text{su}}](\mathbb{1}).$$

The co-associativity of Δ at $\mathbb{1}$ of course means that $LHS = RHS$. An easy calculation shows that the only term of LHS of the form $\mathbb{1} \otimes \text{something}$ is

$$\mathbb{1} \otimes (B \otimes B + \mathbb{1} \otimes B + B \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1})$$

while the only term of RHS of the same form is

$$\mathbf{1} \otimes (B \otimes B + \mathbf{1} \otimes \mathbf{1}).$$

Associativity $RHS = LHS$ then evidently means

$$\mathbf{1} \otimes B + B \otimes \mathbf{1} = 0,$$

which, since $char(\mathbf{k}) \neq 2$, clearly implies $B = 0$. Using the same trick we see also that $A = 0$, therefore $\delta(\mathbf{1})$ must be of the form

$$\delta(\mathbf{1}) = \sum_i J_{(1)}^i \otimes J_{(2)}^i.$$

Since δ is a chain map, trivial on $\mathbf{1}$ and $\mathbf{1}$, $\partial\delta(\mathbf{1}) = 0$, which means that

$$0 = \partial\delta(\mathbf{1}) = \sum_i \partial J_{(1)}^i \otimes J_{(2)}^i - \sum_i J_{(1)}^i \otimes \partial J_{(2)}^i.$$

Looking separately at the components of bidegrees $(1, 0)$ and $(0, 1)$, respectively, and assuming, without loss of generality, that the elements $J_{(1)}^i$ (resp. $J_{(2)}^i$) are linearly independent, we conclude that $\partial J_{(1)}^i = \partial J_{(2)}^i = 0$. Because each cycle in $\mathcal{A}_1(4)$ is a scalar multiple of $\partial(\mathbf{1})$, we see that

$$\delta(\mathbf{1}) = \alpha(\partial(\mathbf{1}) \otimes \partial(\mathbf{1}))$$

for some scalar $\alpha \in \mathbf{k}$. So we managed to cut 35 parameters in (24) to one! Now

$$LHS = \alpha\{\Delta^{\text{su}}(\partial(\mathbf{1})) \otimes \partial(\mathbf{1}) + \partial(\mathbf{1}) \otimes \partial(\mathbf{1}) \otimes \mathbf{1}\} + (\Delta^{\text{su}} \otimes \mathbf{1})\Delta^{\text{su}}(\mathbf{1})$$

and

$$RHS = \alpha\{\partial(\mathbf{1}) \otimes \Delta^{\text{su}}(\partial(\mathbf{1})) + \mathbf{1} \otimes \partial(\mathbf{1}) \otimes \partial(\mathbf{1})\} + (\mathbf{1} \otimes \Delta^{\text{su}})\Delta^{\text{su}}(\mathbf{1}).$$

The only terms of the LHS of the form $\mathbf{1} \otimes something$ are

$$\alpha\{\mathbf{1} \otimes \mathbf{1} \otimes \partial(\mathbf{1}) + \mathbf{1} \otimes \partial(\mathbf{1}) \otimes \mathbf{1}\}$$

while in the RHS , there is only one term of this form, namely

$$\alpha(\mathbf{1} \otimes \Delta^{\text{su}}(\partial(\mathbf{1}))).$$

The only term of the form $\mathbf{1} \otimes \mathbf{1} \otimes something$ in the above two displays is

$$\alpha(\mathbf{1} \otimes \mathbf{1} \otimes \partial(\mathbf{1}))$$

coming from the first term of the first display. This implies that $\alpha = 0$, therefore $\delta = 0$ and $\Delta = \Delta^{\text{su}}$. But this is not possible, because the co-associativity of Δ^{su} is violated already on $\mathbf{1}$, as we saw in (21). \blacksquare

7. Remaining proofs.

In this section we prove Propositions 6 and 10. Let us start with a

Proof of Proposition 6. By definition of ∂_K ,

$$\begin{aligned}\partial_K(c(n), 1) &= \sum_{\substack{r+s=n+1 \\ 1 \leq i \leq r}} (c(r) \circ_i c(s), e) = \sum_{\substack{r+s=n+1 \\ 1 \leq i \leq r}} (-1)^{r(s-2)+i(s+1)} (c(r), 1) \circ_i (c(s), 1) \\ &= \sum_{\substack{r+s=n+1 \\ 1 \leq i \leq r}} (-1)^{(r+i)s+i} (c(r), 1) \circ_i (c(s), 1).\end{aligned}$$

The sign comes from formula (5) setting $l = s - 2$, since $c(s) \in \mathcal{C}_{s-2}(K_s)$. Applying q gives

$$(25) \quad q(\partial_K(c(n), 1)) = \sum_{\substack{r+s=n+1 \\ 1 \leq i \leq r}} (-1)^{(r+i)s+i} q(c(r), 1) \circ_i q(c(s), 1).$$

The expression on the right is a sum over all binary rooted planar metric trees with n leaves and one non-metric edge. On the other hand,

$$\partial_W(q(c(n), 1)) = \sum_{T \in mBin(n)} \partial_W(T, \omega_T).$$

According to (8), the terms in $\partial_W(T, \omega_T)$ are of two types:

Type A, in which a metric edge has been changed to a non-metric edge and

Type B, in which a metric edge has been collapsed, creating a fully metric tree which is binary except for one tertiary vertex.

In the sum of type B terms the same cell appears twice with opposite signs, since there are exactly two binary trees which give rise to the same tree with a unique tertiary vertex. The terms of type A, with one non-metric edge, run over the set of all binary rooted planar metric trees with one non-metric edge, which is the same set as that appearing in the sum on the right of equation (25). It only remains to compare the orientations of the corresponding terms on the two sides of (10). According to Definition 5,

$$(26) \quad \begin{aligned}(-1)^{(r+i)s+i} q(c(r), 1) \circ_i q(c(s), 1) &= \\ (-1)^{(r+i)s+i} (\bar{b}(r) \circ_i \bar{b}(s), e_1 \wedge \cdots \wedge e_{r-2} \wedge f_1 \wedge \cdots \wedge f_{s-2}) &+ \cdots,\end{aligned}$$

where the term shown explicitly on the right is the leading order term relative to the order relation on binary trees, e_1, \dots, e_{r-2} label the edges in $\bar{b}(r)$ and f_1, \dots, f_{s-2} label the edges in $\bar{b}(s)$. Since the definition of the standard orientation on an arbitrary fully metric binary tree involves the same associativities independent of the size of the tree, it is sufficient to compare the orientation of the leading order term in (26) with the orientation of the corresponding term in $\partial_W(q(c(n), 1))$. If these orientations agree, so will the orientations of all the other terms.

Assume $i < r$. Applying ∂_W to the fully metric binary tree with standard orientation appearing in Figure 5, we get (among others) the term $(-1)^{(i+s)} (\bar{b}(r) \circ_i \bar{b}(s), e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_{n-2})$

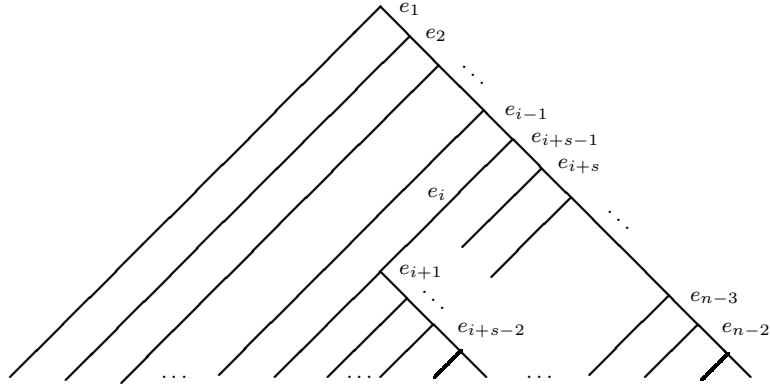


Figure 5: The binary tree in the figure is derived from the maximal binary tree $\bar{b}(n)$ by moving the $s - 1$ adjacent vertices between edges e_i and e_{i+s-1} ; therefore, its standard orientation is $(-1)^{s-1} e_1 \wedge \cdots \wedge e_{n-2}$.

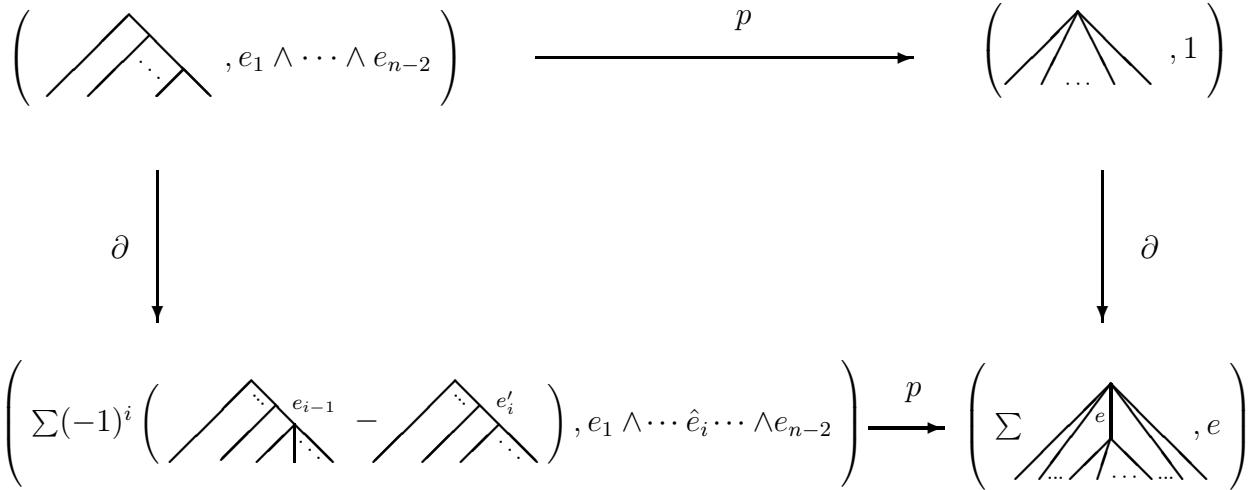


Figure 6: The commutative diagrams for the chain map p in degrees $n - 2$ acting on the maximal binary metric tree.

with e_i changed to a non-metric edge, and the edges labeled $e_{i+1}, \dots, e_{i+s-2}$ corresponding to the edges in $\bar{b}(s)$. Reordering the terms in the orientation element appearing in (26) so that f_1, \dots, f_{s-2} appear in sequence between e_{i-1} and e_i introduces a sign factor $(-1)^{(s-2)(r-i-1)}$. But $(-1)^{(s-2)(r-i-1)}(-1)^{(r+i)s+i} = (-1)^{i+s}$, since so the signs agree. For $i = r$, when $\bar{b}(r) \circ_r \bar{b}(s) = \bar{b}(r+s-1)$, the analysis is much easier and we leave it to the reader. \blacksquare

Proof of Proposition 10. The case $n = 2$ is trivial. Assuming the proposition is true for fully metric trees $(T, \omega_T) \in \mathcal{C}_*(W_m)$ for $m < n$, we will prove it for $\mathcal{C}_k(W_n)$, starting with $k = n - 2$ and descending. In the case $\mathcal{C}_{n-2}(W_n)$, which involves binary fully metric trees, we begin with the maximal binary metric tree. We need to prove the commutativity of Figure 6, which follows from the equations in Figure 7 once we check the signs.

Let us start with the second equation in Figure 7. The tree in parentheses on the left

$$\begin{aligned}
p \left(\begin{array}{c} \text{tree with } e_{i-1} \text{ at root} \end{array} \right) &= \sum_{1 \leq s \leq i+1} \begin{array}{c} \text{tree with } e \text{ at root} \\ \text{underbrace } s \text{ leaves} \\ \text{underbrace } (i+1) \text{ leaves} \end{array} \\
p \left(\begin{array}{c} \text{tree with } e'_i \text{ at root} \end{array} \right) &= \begin{array}{c} \text{tree with } e \text{ at root} \end{array}
\end{aligned}$$

Figure 7: Each of the trees labeling the faces of K_n appears precisely once as a term in $\partial\bar{b}(n)$. The orientation elements (not shown in the figure) are $(-1)^{i-1}e_1 \wedge \cdots \wedge \hat{e}_i \cdots \wedge e_{n-2}$ in the upper left, $(-1)^i e_1 \wedge \cdots \wedge \hat{e}_i \cdots \wedge e_{n-2}$ in the lower left, and e for both trees on the right.

with orientation element $-(-1)^i e_1 \wedge \cdots \wedge \hat{e}_i \cdots \wedge e_{n-2}$, which corresponds to one of the terms appearing in $\partial_W(\bar{b}(n), \omega_{\bar{b}(n)})$, is equal to $-(-1)^i (\bar{b}(i+1), \omega_{\bar{b}(i+1)}) \circ_{i+1} (\bar{b}(n-i), \omega_{\bar{b}(n-i)})$ and therefore its image under p is

$$\begin{aligned}
-(-1)^i p(\bar{b}(i+1), \omega_{\bar{b}(i+1)}) \circ_{i+1} p(\bar{b}(n-i), \omega_{\bar{b}(n-i)}) &= \\
&= -(-1)^i (c(i+1), 1) \circ_{i+1} (c(n-i), 1) \\
&= -(-1)^{i+(i+1)(n-i-2)+(i+1)(n-i+1)} (c(i+1) \circ_{i+1} c(n-i), e) \\
&= (c(i+1) \circ_{i+1} c(n-i), e),
\end{aligned}$$

as required. The orientation element for the trees on the right side of the first equation in Figure 7 with s leaves is

$$(-1)^{i+(n-3)(n-4)/2} e_1 \wedge \cdots \wedge \hat{e}_i \cdots \wedge e_{n-2} \lrcorner (-1)^{s-1} e_1 \wedge \cdots \wedge e_{i-s+1} \wedge e \wedge e_{i-s+2} \wedge \cdots \wedge e_{n-2} = e.$$

Thus p commutes with ∂ on the maximal binary fully metric tree.

Next we will show that p commutes with ∂ for all binary fully metric trees. To simplify notation, we will not indicate the orientation element. For a non-maximal fully metric binary tree T , $p(T) = 0$, because $T_{\min} = T$ has less than $n - 2$ left leaning edges. The only binary fully metric trees for which $p(\partial T) \neq 0$ are trees of the type appearing in Figure 5 with only one right-leaning internal edge.

Let $T^{i,s}$ be the tree in Figure 5, and $T_j^{i,s}$ the term in $\partial T^{i,s}$ with edge e_j non-metric. Then, for $i \neq j$, $T_j^{i,s}$ is a \circ -composition of two fully metric binary trees, one of which is not maximal. Since p is a operad map, the image $p(T_j^{i,s})$ is also a \circ -composition, but one of the two components is zero, since $p(T) = 0$ when T is fully metric binary but not maximal.

For $j \neq i, i-1$ we also have $p(T^{i,s}/e_j) = 0$, because the binary tree $(T^{i,s})_{\min}$ has two right leaning edges. Thus the only terms in $\partial T^{i,s}$ whose image under p is not zero are $T^{i,s}/e_{i-1}$, $T^{i,s}/e_i$, and $T_i^{i,s}$. It follows immediately from the definition of p that

$$(27) \quad p(T^{i,s}/e_i) = p(T_i^{i,s}) + p(T^{i,s}/e_{i-1}).$$

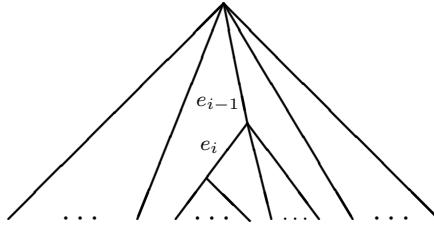
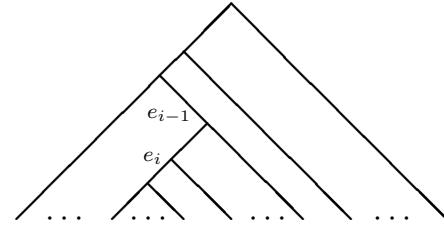
The relevant subtree of T The subtree of both T_{min} and $(T/e_i)_{min}$ 

Figure 8: The only configuration of edges in T which is relevant in the calculation of $p(\partial T)$. The configuration may occur at any vertex, not necessarily at the root.

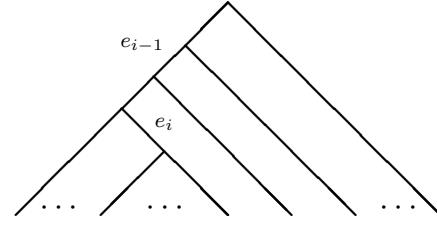
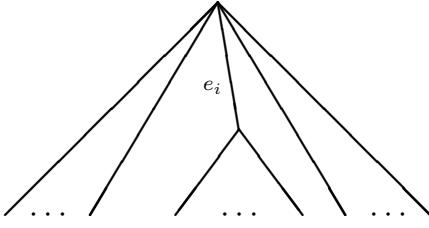


Figure 9: Collapsing the edge e_{i-1} from the subtree on the left of Figure 8 creates the subtree S shown here on the left. Filling-in to get a binary subtree S_{min} creates the subtree of $(T/e_{i-1})_{min}$ shown here on the right, with edge e_i left-leaning and e_{i-1} right-leaning

In fact, the one term appearing in $p(T^{i,s}/e_i)$ and not appearing in $p(T^{i,s}/e_{i-1})$ is $p(T_i^{i,s})$. Therefore,

$$p(\partial T^{i,s}) = p((-1)^{(i-1)}T^{i,s}/e_{i-1} + (-1)^{i-1}T_i^{i,s} + (-1)^i T^{i,s}/e_i) = 0 = \partial p(T^{i,s}).$$

This completes the proof of (13) for $T \in \mathcal{C}_{n-2}(W_n)$.

Now, assuming that (13) is true for all $T \in \mathcal{C}_j(W_n)$ for $k < j \leq n-2$ for all $T \in \mathcal{C}_*(W_m)$ for $m < n$, we need to prove it for $T \in \mathcal{C}_k(W_n)$. Let T be a fully metric tree with k edges labeled e_1, \dots, e_k and T_{min} the binary tree given by filling in, and label the k edges in T_{min} corresponding to the original edges by the same labels. All the other edges of T_{min} are right leaning. If less than $k-1$ of the edges e_1, \dots, e_k in T_{min} are left-leaning, then $p(T) = 0 = p(\partial T)$ and therefore, $\partial p(T) = p(\partial T)$. Suppose first that T_{min} has $k-1$ left-leaning edges, and e_i is right-leaning. Just as for the binary metric trees, the only tree in ∂T for which the image under p is non-zero are $T/e_{i-1}, T/e_i$ and T_i , where e_{i-1} and e_i are adjacent edges in T . The configuration is illustrated in Figure 8. The subtree on the right of Figure 8 appears as a subtree in both in T_{min} and $(T/e_i)_{bin}$, and the tree on the right of Figure 9 appears as a subtree in $(T/e_{i-1})_{min}$. The following equation analogous to (27) applies in this case

$$(28) \quad p(T/e_i) = p(T_i) + p(T/e_{i-1}).$$

and the remainder of the proof of (13) in this case is the same as before.

Next we consider the case when there are exactly k left-leaning edges in T_{min} . In general, for any k cell T such that T_{min} has k left-leaning edges we can choose a $k+1$ cell T^+ such

that ∂T^+ contains T as a summand and all other summands of the type $R_i := T^+/e_i$ for $i = 1, \dots, k$ have the property that $(R_i)_{\min}$ has $k-1$ left-leaning edges. The tree T^+ can be defined as follows: pick any non-binary vertex v in T with $r \geq 3$ incoming edges and replace the corolla with vertex v by subtree of type $c(2) \circ_1 c(r-1)$ with the new edge labeled e_{k+1} . Then $T^+/e_{k+1} = T$ and for $i = 1, \dots, k$, $(T^+/e_i)_{\min}$ has $k-1$ left-leaning edges, since the edge corresponding to e_{k+1} in $(T^+/e_i)_{\min}$ is right-leaning. Denote the faces of type 1 T_j^+ in which a metric edge is changed to a non-metric edge by S_j , $j = 1, \dots, k+1$. By the operad morphism property and the induction assumption we know that (13) is true for each S_j .

Lemma 14. *The validity of (13) for the faces R_i , $i = 1, \dots, k$, implies its validity for T .*

Proof. By definition of R_i and S_j and the property $\partial^2 = 0$,

$$\begin{aligned}\partial T^+ &= T + \sum R_i + \sum S_j, \\ 0 = \partial \partial T^+ &= \partial T + \sum \partial R_i + \sum \partial S_j.\end{aligned}$$

Therefore,

$$-\partial T = \sum \partial R_i + \sum \partial S_j.$$

Moreover,

$$p(\partial R_i) = \partial p(R_i) \quad \text{and} \quad p(\partial S_j) = \partial p(S_j).$$

Thus

$$\begin{aligned}p(-\partial T) &= \sum p(\partial R_i) + \sum p(\partial S_j) = \sum \partial p(R_i) + \sum \partial p(S_j) = \partial p(\sum R_i + \sum S_j) \\ &= \partial p(\partial T^+ - T) = \partial p(\partial T^+) - \partial p(T) \partial \partial(T^+) - \partial p(T) = -\partial p(T).\end{aligned}$$

The summations in the above displays are taken over $1 \leq i \leq k$ and $1 \leq j \leq k+1$. This completes the proof of Lemma 14 and the induction in the proof of (13). \blacksquare

References.

- [1] J.M. Boardman and R.M. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces*. Springer-Verlag, 1973.
- [2] M.R. Gaberdiel and B. Zwiebach. Tensor constructions of open string theories I: Foundations. *Nucl. Phys. B*, 505(3):569–624, Nov. 1997.
- [3] E. Getzler and J.D.S. Jones. Operads, homotopy algebra, and iterated integrals for double loop spaces. Preprint [hep-th/9403055](#), March 1994.
- [4] M. Kontsevich and Y. Soibelman. Homological mirror symmetry and torus fibrations. Preprint [math.SG/0011041](#), November 2000.
- [5] M. Markl. A cohomology theory for $A(m)$ -algebras and applications. *J. Pure Appl. Algebra*, 83:141–175, 1992.
- [6] M. Markl. Models for operads. *Comm. Algebra*, 24(4):1471–1500, 1996.
- [7] M. Markl. Homotopy algebras are homotopy algebras. Preprint [math.AT/9907138](#), to appear in *Forum Mathematicum*, July 1999.

- [8] M. Markl, S. Shnider, and J. D. Stasheff. *Operads in Algebra, Topology and Physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, 2002.
- [9] S. Saneblidze and R. Umble. A diagonal on the associahedra. Preprint [math.AT/0011065](https://arxiv.org/abs/math/0011065), November 2000.
- [10] J.-P. Serre. Homologie singuli  re des espaces fibre  s. *Ann. of Math.*, 54:425–505, 1951. In French.